Example:

Consider $u = u(x,t)$ such that

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \quad \text{for } t>0 \text{ and } x \in (0, 2\pi)$$

$$u(x,0) = h(x) \quad (\text{initial condition})$$

$$u(0,t) = u(2\pi, t) \quad (\text{Boundary condition})$$

Partition $(0, 2\pi)$ into $N$ subintervals that $x_j = j\frac{2\pi}{N}$ $\left(h = \frac{2\pi}{N}\right)$

Discretize the PDE into:

$$\frac{u(x_k, t_{j+1}) - u(x_k, t_j)}{\Delta t} = \frac{u(x_{k+1}, t_j) - 2u(x_k, t_j) + u(x_{k-1}, t_j)}{h^2} + \frac{u(x_{k+1}, t_j) - u(x_{k-1}, t_j)}{2h}$$

for $k = 0, 1, \cdots, N-1$.

Denote $u(x_k, t_j)$ by $u_{k,j}$ and let $\vec{u_j} = \begin{pmatrix} u_{1,j} \\ u_{2,j} \\ \vdots \\ u_{k,j} \\ \vdots \end{pmatrix}$

Then, $\frac{1}{\Delta t}\left(\overrightarrow{u_{j+1}} - \vec{u_j}\right) = D_1 \vec{u_j} + D_2 \vec{u_j}$.

$$D_1 = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & \cdots & 1 \\ 1 & -2 & 1 & \cdots \\ & \ddots & \ddots & \ddots \\ 1 & \cdots & 1 & -2 \end{pmatrix}, \quad D_2 = \frac{1}{2h}\begin{pmatrix} 0 & 1 & \cdots & -1 \\ -1 & 0 & 1 & \cdots \\ & \ddots & \ddots & \ddots \\ 1 & & -1 & 0 \end{pmatrix}$$

Suppose $\vec{u_j} = \sum_{k=0}^{N-1} a_k(j)\, \overrightarrow{e^{ikx}}$.

Then, we have $\frac{1}{\Delta t}\left(\sum_{k=0}^{N-1} \left(a_k(j+1)\overrightarrow{e^{ikx}} - a_k(j)\overrightarrow{e^{ikx}}\right)\right) = \sum_{k=0}^{N-1} a_k(j)\left(D_1 \overrightarrow{e^{ikx}} + D_2 \overrightarrow{e^{ikx}}\right)$

$$= \sum_{k=0}^{N-1} a_k(j)\left(\frac{-4\sin^2\frac{kh}{2}}{h^2}\overrightarrow{e^{ikx}} + \frac{i\sin(hk)}{h}\overrightarrow{e^{ikx}}\right)$$

We can calculate $a_k(0)$ based on the value of $h(x)$.

Then, iteratively, we can solve for $a_k(j)$ for $j = 1, 2, \cdots$.

## FFT for 2D DFT.

Given a vector $(f_k)_{k=0,\cdots,N-1}$, the (1D) DFT of it is

$$\hat{f}(m) = \frac{1}{N} \sum_{k=0}^{N-1} f(k) e^{-2\pi j \frac{mk}{N}}, \quad j = \sqrt{-1}.$$

Given an matrix $(g_{kl})_{k=0,\cdots,M-1}^{l=0,\cdots,N-1}$, the (2D) DFT of it is:

$$\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l) e^{-2\pi j \left(\frac{km}{M} + \frac{ln}{N}\right)}.$$

How can we apply FFT for 2D FFT.?

Simple fact: 2D DFT is separable.

(2D DFT = 2 x 1D DFT).

$$\begin{aligned}
\hat{g}(m,n) &= \frac{1}{MN} \sum_{k=0}^{M-1} \sum_{l=0}^{N-1} g(k,l) e^{-2\pi j \left(\frac{km}{M} + \frac{ln}{N}\right)} \\
&= \frac{1}{MN} \sum_{k=0}^{M-1} \left( \sum_{l=0}^{N-1} g(k,l) e^{-2\pi j \frac{ln}{N}} \right) e^{-2\pi j \frac{km}{M}}
\end{aligned}$$

First calculate $\sum_{l=0}^{N-1} g(k,l) e^{-2\pi j \frac{ln}{N}}$ using 1D FFT.

After that, write $f(k) = \sum_{l=0}^{l-1} g(k,l) e^{-2\pi j \frac{ln}{N}}$.

Then, $\hat{g}(m,n) = \frac{1}{MN} \sum_{k=0}^{M-1} f(k) e^{-2\pi j \frac{km}{M}}$.

Again, we can use 1D FFT to calculate $\hat{g}(m,n)$.

Jacobi iteration:

We aim to solve the linear system $Ax = b$.

Write $A = P - (P - A)$.

Then $Ax = b \iff \left(P - (P-A)\right)x = b \iff Px = (P-A)x + b$

If a iterative scheme satisfies

$$Px_{k+1} = (P-A)x_k + b.$$

If the sequence $x_k \to x^*$,

then $x^*$ satisfies $Px^* = (P-A)x^* + b \iff Ax^* = b$.

In Jacobi iteration, we choose $P = D$, where $D$ is the diagonal part of $A$.

$$Dx_{k+1} = (D-A)x_k + b.$$

If $D$ is invertible, $x_{k+1} = D^{-1}(D-A)x_k + D^{-1}b$, given $x_0$.

Whewher $(x_k)$ is convergent is crucial.

Thm: Jacobi iteration is convergent iff the spectral radius of $D^{-1}(D-A)$ is strictly less than 1.

Def: Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$. Then its spectral radius $\rho(A)$ is defined as

$$\rho(A) := \max\{|\lambda_1|, \cdots, |\lambda_n|\}$$

Write $D^{-1}(D-A) = M$, $\quad D^{-1}b = f$.

Then, $\quad x_{k+1} = M x_k + f$.

Also, $\quad x^* = M x^* + f$.

$$\Rightarrow \underbrace{x_{k+1} - x^*}_{e_{k+1}} = M(\underbrace{x_k - x^*}_{e_k})$$

So, $e_{k+1} = M e_k \iff e_k = M^k e_0$.

Some motivation: if $|M^k e_0| \to 0$, then $x_k \to x^*$.

Let $x$ be an eigenvector of $M$ w.r.t $\lambda$,

then $M^k x = \lambda^k x$.